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Simple Example of Quantum Causal Structure

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The axioms of quantum causal structure and the definition of Alexandrov $\mathbb{T}$-structure are presented in an improved form and an illustrative example is shown.

1. INTRODUCTION

The common ground of all attempts to construct a quantum theory of gravitation is a "field theory" of several geometric variables describing the *space-time structure.* These field theories are usually nonlinear; therefore, a series of difficulties arises with respect to the quantization.

We suggested quite another approach to the quantization of *space-time structure* in an earlier paper (Szabó, 1986), in which the axioms of a *quantum causal structure* were presented. We recall these axioms in an improved form. By way of illustration a simple example is also shown.

The most important consequence of quantum theory is the realization that the logical structure of physical events (that is, the *lattice* structure on which the *probability* theory of quantum physics is based) is *not* Boolean. The classical Boole-lattice-based probability theory has been superseded by a new one based on a more general non-Boolean lattice of events.

The quantization means a procedure by which the Boolean lattice of physical events is exchanged for a non-Boolean lattice of events, namely for the subspace lattice of a Hilbert space. In this sense the quantum theory is "a quantized theory of probability."

It was an important step in the quantization of the basic ideas of mathematical physics when Marlow (1980) proposed to replace the subset lattice in the definition of topology by quantum lattice.

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The *physical events* can be identified with the *subsets of space-time*, namely there is a one-to-one correspondence between the *dual* lattice of physical events and the lattice of space-time subsets (Szabó, 1986).

In contradiction to classical physics, this correspondence *cannot be correct* in quantum theory, because the quantum lattice of physical events is not Boolean; consequently, the dual lattice is not Boolean either. A possible resolution of this inadequacy is if *the whole space-time structure is built upon the ground of quantum lattice of physical events.* As an initial effort, we quantize the causal structure of space-time according to the above conception.

2. THE ROLE OF SUBSET LATTICE IN KRONHEIMER–PENROSE AXIOMS

Recall the axioms of Kronheimer and Penrose (1967) of a causal structure. The quadruple $(X, <_c, \ll, \rightarrow)$ is a *causal structure* if X is an underlying set and $<_c, \ll, \rightarrow$ are relations on X (called *causality, chronology,* and *horismos*) satisfying the following:

K1. $x <_c x$
K2. If $x <_c y$ and $y <_c z$, then $x <_c z$
K3. From $x <_c y$ and $y <_c x$ it follows that $x = y$
K4. Not $x \ll x$
K5. If $x \ll y$, then $x <_c y$
K6. If $x <_c y$ and $y \ll z$, then $x \ll z$
K7. If $x \ll y$ and $y <_c z$, then $x \ll z$
K8. $x \rightarrow y$ if and only if $x <_c y$ and not $x \ll y$
K9. If x_1, x_2, y_1, y_2 are distinct points and $x_i \rightarrow y_j$ for each i and j, then $\rightarrow$ orders x_1, x_2 if and only if it orders y_1, y_2

The last property (called regularity) is not necessary, but it is quite natural, since it holds whenever X is an n-dimensional manifold equipped with a pseudo-Riemann metric of signature $2-n$ (Kronheimer and Penrose, 1967).

The *causal* future set and the *chronological* future set are defined, respectively, as the sets

$$J^+(A) := \{x \in X \mid \text{there exists } a \in A \text{ such that } a <_c x\}$$

$$I^+(A) := \{x \in X \mid \text{there exists } a \in A \text{ such that } a \ll x\}$$

Similarly, the past sets are defined as follows:

$$J^-(A) := \{x \in X \mid \text{there exists } a \in A \text{ such that } x <_c a\}$$

$$I^-(A) := \{x \in X \mid \text{there exists } a \in A \text{ such that } x \ll a\}$$

The *null* future and past sets are defined as

$$C^{\pm}(A) = J^{\pm}(A) \backslash I^{\pm}(A) \tag{1}$$

The future and past sets have the following properties:

P1. $A \subset J^{\pm}(A)$
P2. $I^{\pm}(A) \subset J^{\pm}(A)$
P3. If $(y) \subset J^{\pm}(\{x\})$ and $\{x\} \subset J^{\pm}(\{y\})$, then $x = y$
P4. $J^{\pm}(J^{\pm}(A)) = J^{\pm}(A)$
P5. $J^{\pm}(A \cup B) = J^{\pm}(A) \cup J^{\pm}(B)$
P6. $J^{\pm}(A \cap B) \subset J^{\pm}(A) \cap J^{\pm}(B)$
P7. $I^{\pm}(A \cup B) = J^{\pm}(A) \cup J^{\pm}(B)$
P8. $I^{\pm}(A \cap B) \subset I^{\pm}(A) \cap I^{\pm}(B)$
P9. $J^{\pm}(I^{\pm}(A)) \subset I^{\pm}(A)$
P10. $I^{\pm}(J^{\pm}(A)) \subset I^{\pm}(A)$
P11. Not $\{x\} \subset I^{\pm}(\{x\})$
P12. $\{x\} \subset J^{+}(\{y\})$ is equivalent to $\{y\} \subset J^{-}(\{x\})$
P13. $\{x\} \subset I^{+}(\{y\})$ is equivalent to $\{y\} \subset I^{-}(\{X\})$
P14. If A_1, A_2, B_1, B_2 are distinct subsets and $B_i \subset C^{\pm}(A_j)$ for each i and j, then $A_1 \not\subset J^{\pm}(A_2)$ and $A_2 \not\subset J^{\pm}(A_1)$ is equivalent to $B_1 \not\subset J^{\pm}(B_2)$ and $B_2 \not\subset J^{\pm}(B_1)$

It will be useful to mention here that relation $\rightarrow$ is automatically *horismotic* (Kronheimer and Penrose, 1967), i.e., it is reflective and whenever $\{x_i\}_{1 \le i \le n}$ is a finite sequence such that $x_i \rightarrow x_{i+1}$ for each $i \neq n$ and h, k are integers satisfying $1 \le h \le k \le n$, then

(i) $x_1 \rightarrow x_n$ implies $x_h \rightarrow x_k$
(ii) $x_n \rightarrow x_1$ implies $x_h = x_k$

Equivalently, we have the following property:

P15. For any finite sequence $\{x_i\}_{1 \le i \le n}$ such that $\{x_{i+1}\} \subset C^{+}(\{x_i\})$ for each $i \neq n$ and h, k are integers satisfying $1 \le h \le k \le n$, then
(I) $\{x_n\} \subset C^{+}(\{x_1\})$ implies $\{x_k\} \subset C^{+}(\{x_h\})$
(II) $\{x_1\} \subset C^{+}(\{x_n\})$ implies $x_k = x_h$

It is obvious that two pairs of maps on the subset lattice $J^{\pm}$, $I^{\pm} : \mathfrak{T}(X) \rightarrow \mathfrak{T}(X)$ satisfying P1–P15 define a causal structure on X via the following relations:

$$x <_c y \quad \text{if and only if} \quad \{y\} \subset J^{+}(\{x\})$$

$$x \ll y \quad \text{if and only if} \quad \{y\} \subset I^{+}(\{x\})$$

$$x \rightarrow y \quad \text{if and only if} \quad \{y\} \subset J^{+}(\{x\}) \backslash I^{+}(\{x\})$$

3. QUANTUM CAUSAL STRUCTURE

Let us now replace the subset lattice $(\mathfrak{T}(X), \cap, \cup)$ by the dual of a quantum lattice $\mathfrak{C}, \wedge, \vee)$. We want to define a *quantum causal structure* via maps $J^{\pm}, I^{\pm}: \mathfrak{C} \to \mathfrak{C}$. The causality and chronology determine a unique horismos in the Boolean case. In the non-Boolean case, however, the expression (1) is meaningless in general. We can replace it by

$$C^{\pm}(A) = J^{\pm}(A) \wedge [I^{\pm}(A)]^{\perp} \tag{2}$$

only if we suppose we have an orthocomplementation $\perp$ on lattice $\mathfrak{C}$ (fortunately, a quantum lattice and, consequently, its dual satisfy this

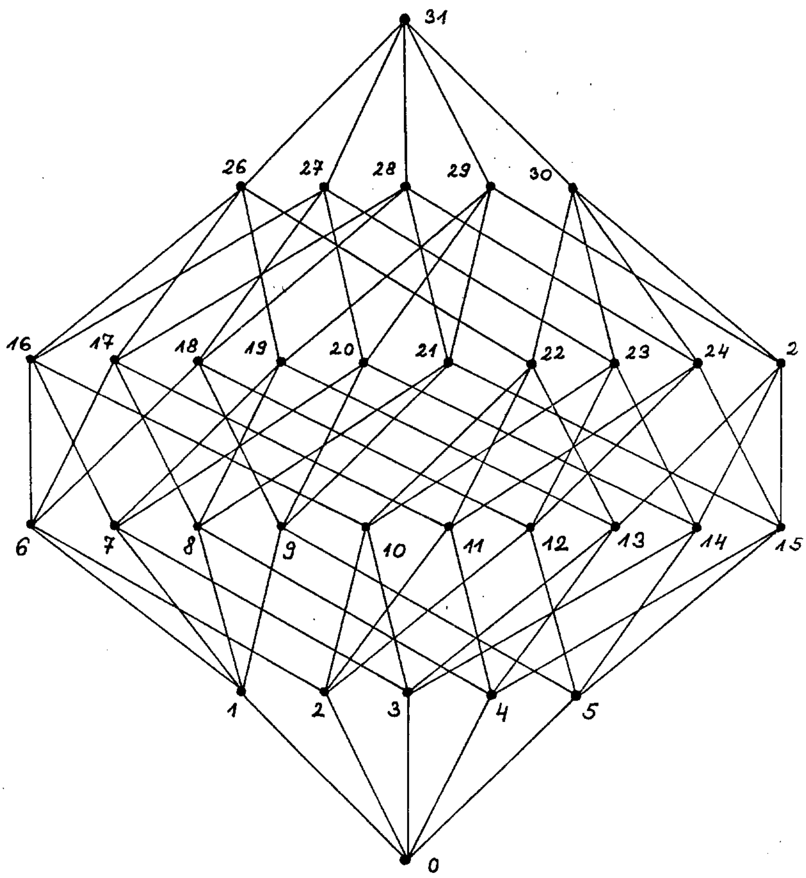

Fig. 1. Boolean lattice with five atoms.

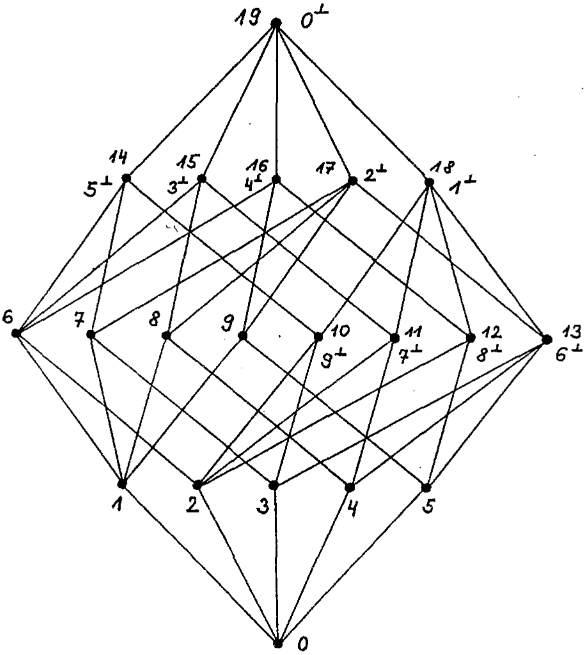

Fig. 2. Non-Boolean quantum lattice with five atoms.

condition). It turns out that the null future and the null past defined in (2) *do not satisfy* condition P15. Therefore we have to *require* this property.

Definition. A *quantum causal structure* is two pairs of maps

$$J^{\pm}: \mathfrak{C} \to \mathfrak{C}$$

$$I^{\pm}: \mathfrak{C} \to \mathfrak{C}$$

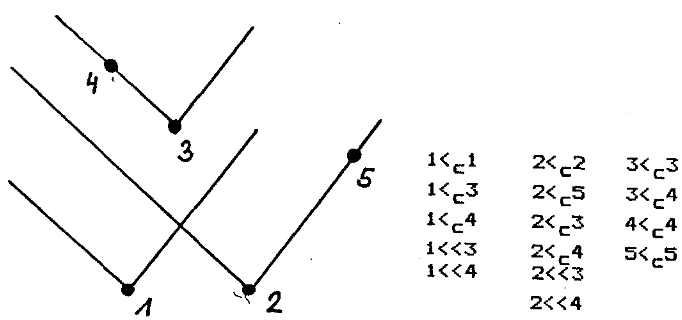

Fig. 3. These relations satisfy the Kronheimer–Penrose axioms, but they cannot be obtained as restriction of a quantum causal structure on non-Boolean lattice $\mathfrak{K}$.

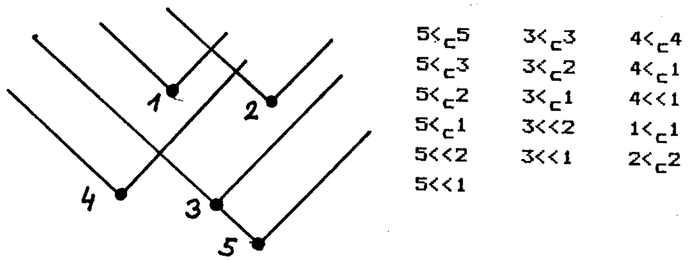

Fig. 4. Relations that are restrictions of a suitable quantum causal structure in case of both $\mathfrak{L}$ and $\mathfrak{K}$.

Table I

A	J^+	I^+	J^-	I^-
1	19	13	1	0
2	30	25	2	0
3	13	0	16	6
4	4	0	26	6
5	5	0	12	0
6	31	25	6	0
7	19	13	16	6
8	19	13	26	6
9	29	13	18	0
10	30	25	16	6
11	30	25	26	6
12	30	25	12	0
13	13	0	26	6
14	25	0	27	6
15	15	0	31	6
16	31	25	16	6
17	31	25	26	6
18	31	25	18	0
19	19	13	26	6
20	19	13	27	6
21	19	13	31	6
22	30	25	26	6
23	30	25	27	0
24	30	25	31	6
25	25	0	31	6
26	31	25	26	6
27	31	25	27	6
28	31	25	27	6
29	29	13	31	6
30	30	25	31	6
31	31	25	31	6

Table II

A	J^+	I^+	J^-	I^-
1	1	0	20	25
2	2	0	23	14
3	16	6	14	0
4	8	1	4	0
5	27	6	5	0
6	6	0	27	25
7	16	6	20	25
8	8	1	29	25
9	27	6	20	25
10	16	6	23	14
11	17	1	30	14
12	27	6	23	14
13	26	6	25	0
14	27	6	14	0
15	31	6	15	0
16	16	6	27	25
17	17	1	31	25
18	27	6	31	25
19	26	6	29	25
20	27	6	20	25
21	31	6	29	25
22	26	6	30	14
23	27	6	23	14
24	31	6	30	14
25	31	6	25	0
26	26	6	31	25
27	27	6	27	25
28	31	6	31	25
29	31	6	29	25
30	31	6	30	14
31	31	6	31	25

with the following properties:

Q1. $A < J^{\pm}(A)$

Q2. $I^{\pm}(A) < J^{\pm}(A)$

Q3. From $x < J^{\pm}(y)$ and $y < J^{\pm}(x)$ it follows that $x = y$

Q4. $J^{\pm}(J^{\pm}(A)) = J^{\pm}(A)$

Q5. $J^{\pm}(A \vee B) = J^{\pm}(A) \vee J^{\pm}(B)$

Q6. $J^{\pm}(A \wedge B) < J^{\pm}(A) \wedge J^{\pm}(B)$

Q7. $I^{\pm}(A \vee B) = I^{\pm}(A) \vee I^{\pm}(B)$

Q8. $I^{\pm}(A \wedge B) < I^{\pm}(A) \wedge I^{\pm}(B)$

Q9. $J^{\pm}(I^{\pm}(A)) < I^{\pm}(A)$

Q10. $I^{\pm}(J^{\pm}(A)) < I^{\pm}(A)$

Table III

A	J^+	I^+	J^-	I^-
1	1	0	17	13
2	2	0	18	13
3	14	6	3	0
4	8	1	4	0
5	16	6	5	0
6	6	0	19	13
7	14	6	17	13
8	8	1	17	13
9	16	6	17	13
10	14	6	18	13
11	15	1	18	13
12	16	6	18	13
13	19	6	13	0
14	14	6	19	13
15	15	1	19	13
16	16	6	19	13
17	19	6	17	13
18	19	6	18	13
19	19	6	19	13

Q11. Not $x < I^{\pm}(x)$

Q12. $x < J^+(y)$ is equivalent to $y < J^-(x)$

Q13. $x < I^+(y)$ is equivalent to $y < I^-(x)$

Q14. If A_1, A_2, B_1, B_2 are distinct elements of $\mathfrak{C}$ and $B_i < C^{\pm}(A_j)$ for each i and j, then $A_1 < J^{\pm}(A_2)$ and $A_2 < J^{\pm}(A_1)$ is equivalent to $B_1 < J^{\pm}(B_2)$ and $B_2 < J^{\pm}(B_1)$.

Table IV

A	$\bar{A}$	A	$\bar{A}$
1	6	11	18
2	2	12	18
3	13	13	13
4	13	14	19
5	13	15	19
6	6	16	19
7	19	17	19
8	19	18	18
9	19	19	19
10	18		

Closed elements: 0, 2, 6, 13, 18, 19.

Open elements: 0, 1, 6, 13, 17, 19.

Q15. For any finite sequence of atoms $\{x_i\}_{1\le i\le n}$ such that $x_{i+1} < C^+(x_i)$ for each $i \neq n$ and h, k are integers satisfying $1 \le h \le k \le n$, then
(I) $x_n < C^+(x_1)$ implies $x_k < C^+(x_h)$
(II) $x_1 < C^+(x_n)$ implies $x_k = x_h$
where A_i, B_i, A, B, $C \in \mathfrak{C}$, x_i, x, $y \in \mathbb{A}(\mathfrak{C})$, $\mathbb{A}(\mathfrak{C})$ denotes the set of atoms of $\mathfrak{C}$, and $C^{\pm}(A) = J^{\pm}(A) \wedge [I^{\pm}(A)]^{\perp}$.

Definition. In a quantum causal structure we define the *causal relations* as follows:

$$\begin{aligned} A <_c B &\quad \text{if and only if} \quad B < J^+(A) \text{ or } A < J^-(B) \\ A \ll B &\quad \text{if and only if} \quad B < I^+(A) \text{ or } A < I^-(B) \\ A \to B &\quad \text{if and only if} \quad B < C^+(A) \text{ or } A < C^-(B) \end{aligned} \tag{3}$$

where $A, B \in \mathfrak{C}$.

If $\mathfrak{C}$ is Boolean lattice, it can be represented by a suitable subset lattice and the quantum causality leads to the usual Kronheimer–Penrose causality on an "underlying set."

4. ALEXANDROV $\mathbb{T}$-STRUCTURE

One can introduce various topologies on a causal space X. The most reasonable of them is the *Alexandrov topology*, i.e., the *coarsest* topology on X in which each $I^+(A)$ and $I^-(A)$ is open.

If $\mathfrak{C}$ is not Boolean, *one cannot define a point set topology* on an "underlying set of causal structure." Fortunately, we can generalize the notion of topology for a non-Boolean lattice. One possible way of introducing topology is to define *the closure* of sets (Kuratowski, 1966). One can introduce a $\mathbb{T}$-structure in a similar way:

Definition. A *closure operation* is a map

$$^-\!: \quad \mathfrak{C} \to \mathfrak{C}, \qquad A \mapsto \bar{A}$$

such that

(i) $A < \bar{A}$
(ii) $\overline{(A \vee B)} = \bar{A} \vee \bar{B}$
(iii) $\bar{\varnothing} = \varnothing$
(iv) $\bar{\bar{A}} = \bar{A}$

Definition. A $\mathbb{T}$*-structure* is a lattice $\mathfrak{C}$ equipped with a closure operation.

On an orthocomplemented lattice this definition is equivalent to that of Marlow (1980).

In a $\mathbb{T}$-structure one can introduce the following basic notions:

Definitions

a. An element $A \in \mathfrak{C}$ *is closed* if and only if $\bar{A} = A$.
b. An element $A \in \mathfrak{C}$ is *open* if and only if $A^{\perp}$ is closed.
c. The *interior* of A, $\mathrm{Int}(A)$, is the largest element of $\mathfrak{C}$ that is smaller than A.
d. The *boundary* of A is $\mathrm{Fr}(A) = \bar{A} \wedge \bar{A}^{\perp}$.

The following definition is a straightforward generalization of the Alexandrov topology:

Definition. The *Alexandrov* $\mathbb{T}$*-structure* is the coarsest $\mathbb{T}$-structure in which each $I^{+}(A)$ and $I^{-}(A)$ is open.

5. REMARKS

The set of atoms $\mathbb{A}(\mathfrak{C})$ can be regarded as a "space-time set." We can restrict causal relations (3) to $\mathbb{A}(\mathfrak{C})$. These restricted relations satisfy Kronheimer–Penrose axioms. There can exist, however, many relations on $\mathbb{A}(\mathfrak{C})$ satisfying Kronheimer–Penrose axioms that cannot be obtained as a restriction of a quantum causal structure on the whole lattice. All this means a considerable restraint for the possible causal relations. The following example illustrates this situation very well.

6. EXAMPLE

Consider two lattices with the same set of atoms. $\mathfrak{L}$ is a Boolean lattice, but $\mathfrak{K}$ is not Boolean (see Figures 1 and 2). For instance, the causal relations shown on Figure 3 satisfy the Kronheimer–Penrose axioms. In the case of Boolean lattice $\mathfrak{L}$ a suitable quantum causal structure is shown in Table I.

However, in the case of non-Boolean lattice $\mathfrak{K}$, from the supposed relations we have $1 < J^{+}(1)$ and $3 < J^{+}(1)$ and $4 < J^{+}(1)$, from which it follows (see Figure 2) that $J^{+}(1) = 17$. Consequently, $1 <_c 5$, which leads to a contradiction.

Another collection of causal relations (shown on Figure 4) can be obtained as restriction of a quantum causal structure in both cases. The quantum causal structure for Boolean lattice $\mathfrak{L}$ is shown in Table II.

For non-Boolean quantum lattice $\mathfrak{K}$ the situation is depicted in Table III. One can verify that Tables II and III satisfy the axioms Q1–Q15.

Finally, construct the Alexandrov $\mathbb{T}$-structure corresponding to the quantum causal structure of Table III on lattice $\mathfrak{K}$. The chronological future and past elements are 0, 1, 6, 13. Therefore, the closure operation has to act identically on $0^{\perp} = 19$, $1^{\perp} = 18$, $6^{\perp} = 13$, $13^{\perp} = 6$. The coarsest $\mathbb{T}$-structure on $\mathfrak{K}$ satisfying this condition is shown in Table IV.

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REFERENCES

Jauch, J. M. (1968). *Foundation of Quantum Mechanics*, Addison-Wesley, Reading, Massachusetts.

Kronheimer, E. H., and Penrose, R. (1967). *Proceedings of the Cambridge Philosophical Society*, **63**, 418.

Kuratowski, K. (1966). *Topology*, Vol. I, Academic Press, New York.

Marlow, A. R. (1980). *International Journal of Theoretical Physics*, **19**, 516.

Szabó, L. (1986). *Journal of Mathematical Physics*, **27**, 2709.